THE ACTION OF A CONCENTRATED FORCE ON AN ECCENTRIC RING

(DEISTVIE SOSREDOTOCHENNOI SILY NA EKSTSENTRICHESKOE KOL'TSO)

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The general solution of the plane problem of the theory of elasticity for a region bounded by two eccentric circles in a plane was given by Jeffer [1] and Weinel [2] in bipolar coordinates. The loading was assumed to be in the form of a trigonometric series. This solution cannot be applied to a case of concentrated loading, since the trigonometric series become divergent.

Sen Gupta [3] analyzed the deformation of an eccentric ring acted upon by two diametrically opposite forces applied along the axis symmetry. However, a general case of forces acting on such a ring was not considered.

Below are given expressions for the stress function and for the components of stress resulting from the action of any concentrated force on the outside periphery of an eccentric ring. It is assumed that the reactions for a given force are stresses acting along the ring's boundaries. These reaction stresses reduce to zero when several forces acting on the ring are in equilibrium.

In the solution of this problem the bipolar coordinates α and β are used. Their relation to the rectangular coordinates x and y is given by the following expressions.*

$$x = \frac{\sinh\alpha}{\cosh\alpha + \cos\beta}, \quad y = \frac{\sin\beta}{\cosh\alpha + \cos\beta} \tag{1}$$

Let us consider the state of stress in an eccentric ring defined by coordinates $\alpha = \alpha_1$ and $\alpha = \alpha_2$; the radii of these circles being respect-

* The information on bipolar coordinates, and on many problems solved in terms of such coordinates, may be found in a book by Ufliand [4].

ively equal to r_1 = cosech α_1 and r_2 = cosech α_2; the eccentricity is equal to $e = |\coth \alpha_1 - \coth \alpha_2|$ (Fig. 1).

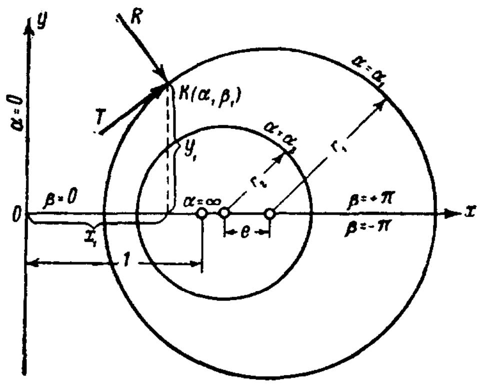

Fig. 1.

Let it be assumed that a force is applied at point K of the periphery $\alpha = \alpha_1$ (the bipolar coordinates of point K are α_1, β_1, and the rectangular coordinates are x_1, and y_1). The applied force has normal component R and tangential component T and is reacted on by the following tangential and normal stresses along both boundaries of the ring:

$$\tau_{\alpha\beta} = \pm \frac{1}{4\pi} (\cosh \alpha + \cos \beta)(Y \operatorname{ch} \alpha + M \sinh \alpha)$$

$$\sigma_\alpha = \pm \frac{1}{4\pi} [X (\cos^2 \beta + 2 \cosh \alpha \cos \beta + 1) - Y \sinh \alpha \sin \beta + M (\cosh \alpha + \cos \beta) \sin \beta] \quad (2)$$

where X and Y are projections of the applied force on the x and y coordinate axes respectively, and $M = Xy_1 - Yx_1$ is the moment of the force about the origin of coordinate axes. (The upper signs are given for the stresses along the outer periphery and lower signs for the stresses along the inner periphery). Consequently the stresses are reduced to zero when

$$\Sigma X = \Sigma Y = \Sigma M = 0.$$

Let the stress function be equal to the sum of two functions, both of which are biharmonic and satisfy single-sign displacement conditions.

$$\varphi = \varphi_1 + \varphi_2 \quad (3)$$

The function ϕ_1 has a singularity at point $K(x_1, y_1)$ of the type

$$\frac{1}{\pi} [-X(y - y_1) + Y(x - x_1) \tan^{-1} \frac{y - y_1}{x - x_1}$$

which corresponds to the application of a force at point K on the ring's periphery; the function is defined by the following formula

$$g\varphi_1=\frac{1}{2\pi}\left\{\mp\frac{1}{2}\beta\left[-X\sin\beta+Y\sinh\alpha+M(\cosh\alpha+\cos\beta)\right]\pm\frac{1}{4}(1-\nu)\alpha\left[X\sinh\alpha+\right.\right.$$

$$\left.\left.+Y\sin\beta\right]+\left[T\sinh t-R\sin\theta+(Tx_1+Ry_1)(\cosh t-\cos\theta)\right]2\tan^{-1}\left(\operatorname{cth}\frac{1}{2}t\tan\frac{1}{2}\theta\right)\right\}\quad(4)$$

$$(g=\cosh\alpha+\cos\beta,\ t=\alpha-\alpha_1,\ \theta=\beta-\beta_1)$$

where ν is Poisson's ratio. In this and in the following formulas the upper sign applied when $\alpha_1<\alpha_2$, i.e., when the force acts on the outer periphery of the ring, and the lower sign applies when $\alpha_1>\alpha_2$, where the force is applied to the inner periphery of the ring. The corresponding stresses may be found by known formulas for the components of stress by means of a stress function expressed in bipolar coordinates (see, for instance, [4] p. 171), as follows:

$$[\tau_{\alpha\beta}]_1=\frac{1}{2\pi}(\cosh\alpha+\cos\beta)\left[-(T\sin\theta+R\sinh t)\frac{\sinh t\sin\theta}{(\cosh t-\cos\theta)^2}+X\sin\beta-\right.$$

$$-Y\sinh\alpha-M(\cosh\alpha+\cos\beta)-(Tx_1+Ry_1)\frac{\cosh t\cos\theta-1}{\cosh t-\cos\theta}\pm$$

$$\left.\pm\frac{1}{2}(Y\cosh\alpha+M\sinh\alpha)\mp\frac{1-\nu}{4}Y\cos\beta\right]$$

$$[\sigma_\beta-\sigma_\alpha]_1=\frac{1}{\pi}(\cosh\alpha+\cos\beta)\left[(T\sin\theta+R\sinh t)\frac{\cosh t\cos\theta-1}{(\cosh t-\cos\theta)^2}-\right.$$

$$\left.-(Tx_1+Ry_1)\frac{\sinh t\sin\theta}{\cosh t-\cos\theta}\mp\frac{1}{2}(X\cos\beta+M\sin\beta)\pm\frac{1-\nu}{4}X\cosh\alpha\right]$$

$$[\sigma_\beta+\sigma_\alpha]_1=\frac{1}{\pi}\left[-(T\sin\theta+R\sinh t)\frac{\cosh\alpha_1+\cos\beta_1}{\cosh t-\cos\theta}+T\sin\beta_1-R\sinh\alpha_1-\right.$$

$$\left.-X\sinh\alpha\cos\beta+Y\cosh\alpha\sin\beta\pm\frac{3-\nu}{4}(X\cosh\alpha\cos\beta+X-Y\sinh\alpha\sin\beta)\right]\quad(5)$$

The function ϕ_2 must remove from the boundary of the ring all stresses not included in equation (2), and may be chosen in the following form of a series:

$$g\varphi_2=\frac{1}{2\pi}\left\{J\alpha(\cosh\alpha+\cos\beta)+C\cos\beta+\sum_{n=1}^{\infty}\left[f_n^c(\alpha)\cos n\beta+f_n^s(\alpha)\sin n\beta\right]\right\}\qquad(6)$$

Here

$$f_n(\alpha)=A_n[\cosh(n+1)t-\cosh(n-1)t]+B_n[(n-1)\sinh(n+1)t-(n+1)\sinh(n-1)t]$$

$$f_1(\alpha)=A_1\cosh 2t+B_1\sinh 2t\qquad(t=\alpha-\alpha_1)\qquad(n\geqslant 2)\qquad(7)$$

The constants in this function must be such as to satisfy boundary conditions. The stresses corresponding to the function ϕ_2 are determined by the following formulas

$$[\tau_{\alpha\beta}]_2 = \frac{1}{2\pi}(\cosh\alpha + \cos\beta)\left\{J\sin\beta + \sum_{n=1}^{\infty}[nf_n^{c'}(\alpha)\sin n\beta - nf_n^{s'}(\alpha)\cos n\beta]\right\} \qquad (8)$$

$$[\sigma_\beta - \sigma_\alpha]_2 = \frac{1}{\pi}(\cosh\alpha + \cos\beta)\left\{J\sinh\alpha + \sum_{n=1}^{\infty}[F_n^c(\alpha)\cos n\beta + F_n^s(\alpha)\sin n\beta]\right\}$$

$$[\sigma_\beta + \sigma_\alpha]_2 = \frac{1}{\pi}\left\{-C + \sum_{n=1}^{\infty}[\Phi_n^c(\alpha)\cos n\beta + \Phi^s(\alpha)\sin n\beta]\right\}$$

Here

$$F_n(\alpha) = nA_n[(n+1)\cosh(n+1)t - (n-1)\cosh(n-1)t] +$$
$$+ n(n^2-1)B_n[\sinh(n+1)t - \sinh(n-1)t] \qquad (n \geqslant 2)$$

$$F_1(\alpha) = 2A_1\cosh 2t + 2B_1\sinh 2t \qquad (9)$$

$$\Phi_n(\alpha) = [(n+1)A_{n+1} + 2n\cosh\alpha_1 A_n + (n-1)A_{n-1}]\cosh nt + [-2\sinh\alpha_1 A_n +$$
$$+ (n+1)(n+2)B_{n+1} + 2(n^2-1)\cosh\alpha_1 B_n + (n-1)(n-2)B_{n-1}]\sinh nt. \qquad (n \geqslant 3)$$

$$\Phi_2(\alpha) = [3A_3 + 4\cosh\alpha_1 A_2 + A_1[\cosh 2t +] - 2\sinh\alpha_1 A_2 + 12B_3 + 6\cosh\alpha_1 B_2 + B_1]\sinh 2$$

$$\Phi_1(\alpha) = [2A_2 + 2\cosh\alpha_1 A_1 - 2\sinh\alpha_1 B_1[\cosh t + [-2\sinh\alpha_1 A_1 + 6B_1 + 2\cosh\alpha_1 B_1]\sinh t$$

In order to determine the constants, the function $g\phi_1$ is also expressed in the form of trigonometric series using expansion

$$\tan^{-1}\left(\coth\frac{t}{2}\tan\frac{\theta}{2}\right) = \pm\left[\frac{1}{2}\theta + \sum_{n=1}^{\infty}\frac{1}{n}e^{\mp nt}\sin n\theta\right] \qquad (10)$$

One obtains

$$g\varphi_1 = \frac{1}{2\pi}\left\{G + H \pm\right.$$
$$\mp \frac{1-\nu}{4}\alpha(X\sinh\alpha + Y\sin\beta) + [\pm R(\cosh\alpha_1 + \cos\beta_1) + R\sinh\alpha_1 \mp {}^3/_2 X]\cos\beta +$$
$$+ (e^{\mp 2t} - 1)[-T\cos\beta_1 \mp {}^1/_2 M]\sin\beta + (e^{\mp 2t} - 1)[T\sin\beta_1 \mp {}^1/_2 X]\cos\beta \pm$$
$$\left.\pm 2T\sum_{n=2}^{\infty}\frac{\sinh t}{n}e^{\mp nt}\sin n\theta + 2\sum_{n=2}^{\infty}[Rn\cos\theta - (Tx_1 + Ry_1)\sin n\theta]\frac{n\sinh t \pm \cosh t}{n(n^2-1)}e^{\mp nt}\right\}$$

where

$$G = \mp\beta_1[-X\sin\beta + Y\sinh\alpha + M(\cosh\alpha + \cos\beta)] \pm$$
$$\pm [R\sin\beta_1 - {}^3/_2 M]\sin\beta + Re^{\pm\alpha_1}\sinh\alpha \mp Re^{\pm\alpha_1}(\cosh a + \cos\beta)$$
$$H = + {}^1/_2\beta[-X\sin\beta + Y\sinh\alpha + M(\cosh\alpha + \cos\beta)]$$

The terms in the expression for G do not affect stresses; thus they need not be considered; the terms in the expression for H, when $\alpha = \alpha_2$, give stresses entering into (2); the stresses when $\alpha = \alpha_2$, which depend on the remaining terms of series given by (10), must be taken care of by means of function ϕ_2.

Having the above in mind, the coefficients of $\cos n\beta$ and $\sin n\beta$ in the sums $g\phi_1 + g\phi_2$ and $\partial(g\phi_1)/\partial\alpha + \partial(g\phi_2)/\partial\alpha$ when $\alpha = \alpha_2$, may be equated to zero, and thus coefficients A_n, $B_n (n \geqslant 2)$ may be determined as follows:

$$A_n{}^c = \mp \frac{\sinh^2 t_1}{\Delta_n}[Rn\cos n\beta_1 + (Tx_1 + Ry_1)\sin n\beta_1] + \frac{T\sin n\beta_1}{n\Delta_n}[n^2\sinh^2 t_1 \mp$$

$$\mp n\sinh t_1\cosh t_1 \pm e^{\mp nt_1}\sinh nt_1]$$

$$B_n{}^c = \frac{1}{n(n^2-1)\Delta_n}[Rn\cos n\beta_1 + (Tx_1 + Ry_1)\sin n\beta_1][n^2\sinh^2 t_1 \pm n\sinh t_1\cosh t_1 \pm$$

$$\pm e^{nt_1}\sinh nt_1] \mp \frac{T\sin n\beta_1}{\Delta_n}\sinh^2 t_1$$

$$A_n{}^s = \mp\frac{\sinh^2 t_1}{\Delta_n}[Rn\sin n\beta_1 - (Tx_1 + Ry_1)\cos n\beta_1] -$$

$$-\frac{T\cos n\beta_1}{n\Delta_n}[n^2\sinh^2 t_1 \mp n\sinh t_1\cosh t_1 \pm e^{\mp nt_1}\sinh nt_1]$$

$$B_n{}^s = \frac{1}{n(n^2-1)\Delta_n}[Rn\sin n\beta_1 + (Tx_1 + Ry_1)\cos n\beta_1]\times$$

$$\times[n^2\sinh^2 t_1 \pm n\sinh t_1\cosh t_1 \pm e^{\mp nt_1}\sinh nt_1] \pm \frac{T\cos n\beta_1}{\Delta_n}\sinh^2 t_1 \qquad (11)$$

$$\Delta_n = \sinh^2 nt_1 - n^2\sinh^2 t_1 \qquad (t_1 = \alpha_2 - \alpha_1)$$

The constants A_1, B_1, C and J may be found by comparing the stresses given by formulas (5) and (8), when $\alpha = \alpha_1$ and $\alpha = \alpha_2$, with those given by (2). Then

$$A_1{}^c = \pm T\frac{e^{\mp 2t_1}\sin\beta_1}{\sinh 2t_1} \pm \frac{1}{2}X + \frac{1}{2}J\tanh t_1, \quad B_1{}^c = -\frac{1}{2}(X + J)$$

$$C = \mp T\frac{e^{\mp 2t_1}\sin\beta_1}{\sinh 2t_1} - R\sinh\alpha_1 \mp \frac{1}{2}X \mp \frac{1-\nu}{4}X\sinh^2\alpha_1 - \frac{1}{2}J(\tanh t_1 + \sinh 2\alpha_1)$$

$$J = \frac{1}{\tanh t_1(\sinh^2\alpha_2 + \sinh^2\alpha_1)}\Big[\pm T\tanh t_1\sin\beta_1 \mp R(\cosh\alpha_1 + \cos\beta_1) \pm \frac{3}{2}X \mp$$

$$\mp\frac{1-\nu}{4}X(\sinh^2\alpha_2 - \sinh^2\alpha_1)\Big]$$

$$A_1{}^s = \mp T\frac{e^{\mp 2t_1}\cos\beta_1}{\sinh 2t_1} \pm \frac{1}{2}M \pm \frac{1-\nu}{8}Y\tanh t_1 \qquad (12)$$

$$B_1{}^s = -\frac{1}{2}M \mp \frac{1-\nu}{8}Y$$

Thus, the stress function for the given problem is determined.

As an example, let us consider the compression of an eccentric ring acted upon by two equal forces applied along a diameter of the outer circumference of the ring, Fig. 2.

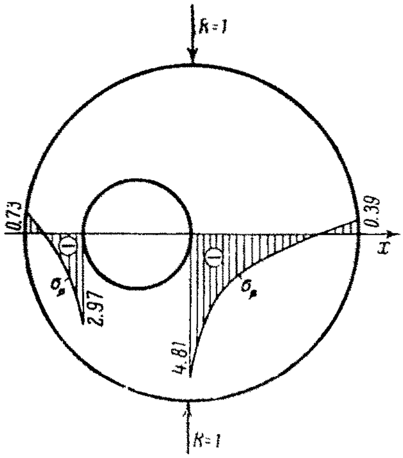

Fig. 2.

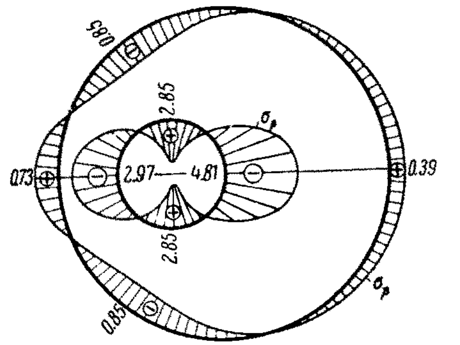

Fig. 3.

The ring is bounded by the curves $\alpha_1 = 1.0$ and $\alpha_2 = 2.0$; the radius of the outer circle is $r_1 = 0.8509$; the radius of the inner circle is $r_2 = 0.2757$; the eccentricity is $e = 0.2757$. The coordinates of a point of application of the force are $\alpha = 1$ and $\beta = \pm 2.2758$. Here, obviously, $\Sigma X = \Sigma Y = \Sigma M = 0$ and the periphery of the ring is free of stresses. For each force $R = 1$ and $T = 0$. The first five terms of the series given by (6) and (7) are used in computations. The following values of coefficients ($\nu = 0.3$) are obtained from formulas given by (11) and (12):

$$A_1^c = -0.06157,\quad A_2^c = 0.42017,\quad A_3^c = -0.09448,\quad A_4^c = +0.01347,\quad A_5^c = -0.00054$$
$$B_1^c = +0.08085,\quad B_2^c = -0.48890,\quad B_3^c = +0.05235,\quad B_4^c = -0.00485,$$
$$B_5^c = +0.00014,\qquad C = -1.99559,\quad J = -0.16170$$

The stresses are computed by formulas (5) for each of the two forces separately and added to stresses computed by formulas (8). Figs. 2 and 3 show diagrams of stresses σ_β along the diameter on the x-axis and along the periphery of the ring.

A half-plane with a round hole may be considered as a special case of an eccentric ring, when the radius of the outer circle is equal to infinity. Thus, all formulas derived in this article may be applied to a problem where a concentrated force is acting on a half-plane with a circular opening; it is only necessary to put $\alpha = 0$ for the straight-line boundary. It is also necessary to consider forces which must act as reactions to a given loading at an infinitely distant point ($\alpha = 0$, $\beta = \pm \pi$) on the plane.

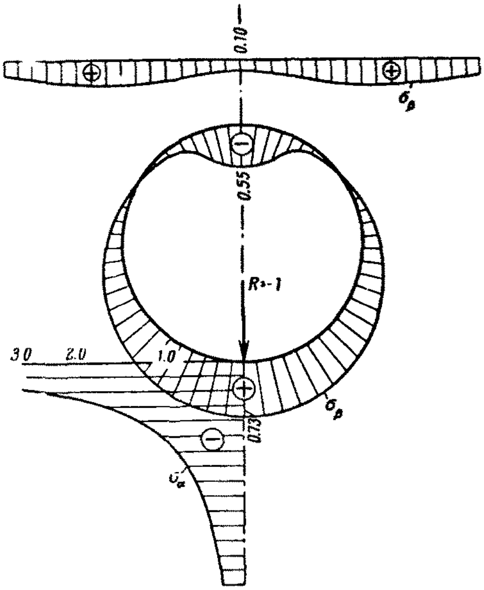

Fig. 4.

Fig. 4 shows a diagram of stresses, σ_β along the periphery of a circular opening and stresses σ_α along the diameter on the x-axis, when a force is applied at a point along the circumference of the opening in an elastic half-plane. The radius of the opening is $r = 0.8509$, the distance from the boundary line of the half-plane to the center of the opening is $d = 1.3130$. The force having components $R = -1$ and $T = 0$ is applied at a point having coordinates $\alpha = 1.0$ and $\beta = \pm \pi$. The reaction is a force, $R = +1$ and $T = 0$, acting at a point having coordinates $\alpha = 0.0$ and $\beta = \pm \pi$.

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